Probability Basics

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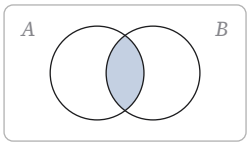
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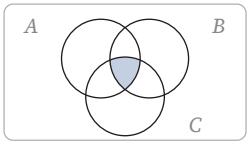
Rules

1 Intersection (And) ∩

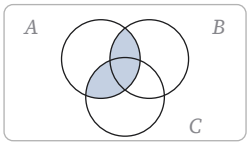
$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$



$$P(A \cap B \cap C) = P(A) + P(B) + P(C) - P(A \cup B) - P(B \cup C) - P(A \cup C) + P(A \cup B \cup C)$$



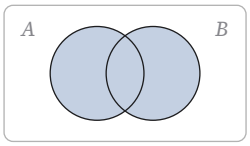
$$P(A \cap (B \cup C)) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$



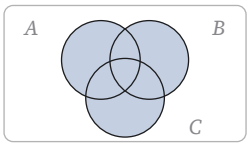
Keywords: *and, both, together*

2 Union (Or) ∪

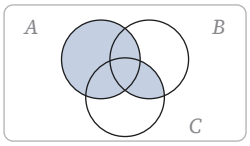
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$



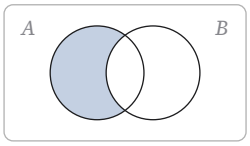
$$P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap B \cap C)$$



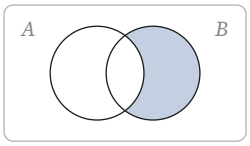
Keywords: *or, either A or B, at least one*

3 Difference (Only) —

$$P(A - B) = P(A) - P(A \cap B)$$



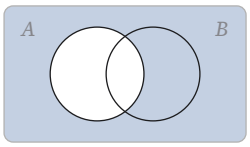
$$P(B - A) = P(B) - P(A \cap B)$$



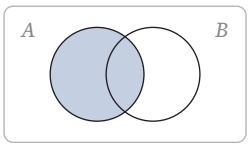
Keywords: *only, but not*

4 Complement (Not) $\overline{A}$ or A' or A^c

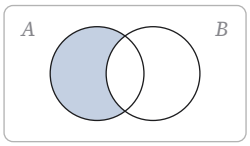
$$P(\overline{A}) = 1 - P(A)$$



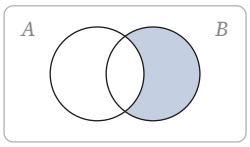
$$P(A) = 1 - P(\overline{A})$$



$$P(A \cap \overline{B}) = P(A - B)$$

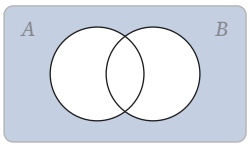


$$P(\overline{A} \cap B) = P(B - A)$$



Neither A nor B

opposite of Either A or B $P(A \cup B)$



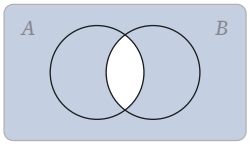
$$P(\overline{A \cup B}) = 1 - P(A \cup B)$$

same

$$P(\overline{A} \cap \overline{B})$$

Not both

opposite of Both A and B $P(A \cap B)$



$$P(\overline{A \cap B}) = 1 - P(A \cap B)$$

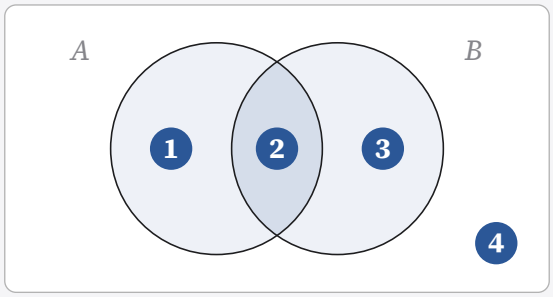
same

$$P(\overline{A} \cup \overline{B})$$

Keywords: *not, non-occurrence, neither*

Warm-up

Which regions does each phrase mean? Write the regions, then the rule.



1 + 2 + 3 + 4 = 1

	Regions	Rule
both · and · together	2	$P(A \cap B)$
A only · A but not B	1	$P(A) - P(A \cap B)$
B only · B but not A	3	$P(B) - P(A \cap B)$
neither A nor B	4	$1 - P(A \cup B)$
A or B · either · at least one	1 + 2 + 3	$P(A \cup B)$
not both · not together	1 + 3 + 4	$1 - P(A \cap B)$
exactly one	1 + 3	$P(A \cup B) - P(A \cap B)$
not A	3 + 4	$1 - P(A)$

Q1

Let A and B be two events of a sample space, with $P(A) = 0.5$, $P(B) = 0.7$ and $P(A \cup B) = 0.9$. Find:

1. The occurrence of both A and B
2. The occurrence of A only
3. The occurrence of B only
4. The non-occurrence of A or B $(\overline{A} \cup \overline{B})$
5. The non-occurrence of A and B together $(\overline{A \cap B})$
6. Neither A nor B
7. Either A or B

Solution

Given: $P(A) = 0.5,\ \ P(B) = 0.7,\ \ P(A \cup B) = 0.9$

1) *The occurrence of both A and B*

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= 0.5 + 0.7 - 0.9 = 0.3$$

2) *The occurrence of A only*

$$P(A - B) = P(A) - P(A \cap B)$$

$$= 0.5 - 0.3 = 0.2$$

from part 1

3) *The occurrence of B only*

$$P(B - A) = P(B) - P(A \cap B)$$

$$= 0.7 - 0.3 = 0.4$$

$P(A \cap B) = 0.3$ *from part 1*

4) *The non-occurrence of A or B* $(\overline{A} \cup \overline{B})$

$$P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B)$$

$$= 1 - 0.3 = 0.7$$ $P(A \cap B) = 0.3$ *from part 1*

5) *The non-occurrence of A and B together* $(\overline{A \cap B})$

$$P(\overline{A \cap B}) = 1 - P(A \cap B)$$

$$= 1 - 0.3 = 0.7$$ $P(A \cap B) = 0.3$ *from part 1*

6) *Neither A nor B*

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$$

$$= 1 - 0.9 = 0.1$$

7) *Either A or B*

$P(A \cup B) = 0.9$ *given in the question*

Q2

Let A and B be two events of a sample space S, with $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \cup B) = 0.7$. Find the probability of B but not A.

Solution

Given: $P(A) = 0.4,\ \ P(B) = 0.5,\ \ P(A \cup B) = 0.7$

First, find $P(A \cap B)$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= 0.4 + 0.5 - 0.7 = 0.2$$

Then, B but not A

$$P(B - A) = P(B) - P(A \cap B)$$

from the first step

$$= 0.5 - 0.2 = 0.3$$

Q3

Let A and B be two events of a sample space S, with $P(A) = 0.6$, $P(B) = 0.3$ and $P(A \cap B) = 0.2$. Find the probability of the non-occurrence of A and B together.

Solution

Given: $P(A \cap B) = 0.2$

Non-occurrence of A and B together

$$P(\overline{A \cap B}) = 1 - P(A \cap B)$$

$$= 1 - 0.2 = 0.8$$

Q4

A couple has two children. Find the probability that both of them are girls.

Solution

List the sample space

$$S = \{BB,\ BG,\ GB,\ GG\}, \quad n(S) = 4$$

The event: both girls

$$E = \{GG\}, \quad n(E) = 1$$

Probability

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}$$

Q5

A single die is rolled. What is the probability that a 3 will show face up?

Solution

List the sample space

$$S = \{1, 2, 3, 4, 5, 6\}, \quad n(S) = 6$$

The event: a 3

$$E = \{3\}, \quad n(E) = 1$$

Probability

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}$$

Q6

Two dice are rolled. What is the probability that a two-spot will show face up?

Solution

Given: $n(S) = 6 \times 6 = 36$ outcomes

The event: a two-spot shows

Shade every box that has the number 2, then count the shaded boxes (red numbers).

First die \ Second die	1	2	3	4	5	6
1	(1, 1)	(1, 2) 1	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1) 2	(2, 2) 3	(2, 3) 4	(2, 4) 5	(2, 5) 6	(2, 6) 7
3	(3, 1)	(3, 2) 8	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2) 9	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2) 10	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2) 11	(6, 3)	(6, 4)	(6, 5)	(6, 6)

$n(E) = 11$

Probability

$$P(E) = \frac{n(E)}{n(S)} = \frac{11}{36}$$

Q7

Two dice are rolled once. Find the probability of getting:

1. A sum of 9
2. A sum of 7 or 11
3. Doubles
4. A sum less than or equal to 9
5. A sum greater than 10
6. The first die greater than the second
7. An even sum
8. Both dice odd

Solution

Given: $n(S) = 6 \times 6 = 36$ *red numbers count the shaded boxes*

1) *A sum of 9*

Shade every box whose two numbers add up to 9.

$$E = \{(3,6), (4,5), (5,4), (6,3)\}$$

$$n(E) = 4$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{36} = \frac{1}{9}$$

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6) *1*
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5) *2*	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4) *3*	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3) *4*	(6, 4)	(6, 5)	(6, 6)

2) *A sum of 7 or 11*

Shade every box whose numbers add up to 7, and every box that adds up to 11.

Sum 7: 6 boxes Sum 11: 2 boxes

$n(E) = 6 + 2 = 8$

$$P(E) = \frac{n(E)}{n(S)} = \frac{8}{36} = \frac{2}{9}$$

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6) 1
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5) 2	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4) 3	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3) 4	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2) 5	(5, 3)	(5, 4)	(5, 5)	(5, 6) 6
6	(6, 1) 7	(6, 2)	(6, 3)	(6, 4)	(6, 5) 8	(6, 6)

3) *Doubles*

Shade every box where both dice show the same number.

$E = \{(1, 1), (2, 2), \ldots, (6, 6)\}$

$n(E) = 6$

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

	1	2	3	4	5	6
1	(1, 1) 1	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2) 2	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3) 3	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4) 4	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5) 5	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6) 6

4) *A sum less than or equal to 9*

Easier with the opposite: shade every box whose sum is 10 or more, then subtract from 1.

Sum ≥ 10: 6 boxes

$$P(\text{sum} \leq 9) = 1 - P(\text{sum} \geq 10)$$

$$= 1 - \frac{6}{36} = \frac{5}{6}$$

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6) 1
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5) 2	(5, 6) 3
6	(6, 1)	(6, 2)	(6, 3)	(6, 4) 4	(6, 5) 5	(6, 6) 6

5) *A sum greater than 10*

Shade every box whose sum is 11 or 12.

$E = \{(5, 6), (6, 5), (6, 6)\}$

$n(E) = 3$

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6) 1
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5) 2	(6, 6) 3

6) *The first die greater than the second*

Shade every box where the first number is bigger than the second.

$$n(E) = 15$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{15}{36} = \frac{5}{12}$$

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1) 1	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1) 2	(3, 2) 3	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1) 4	(4, 2) 5	(4, 3) 6	(4, 4)	(4, 5)	(4, 6)
5	(5, 1) 7	(5, 2) 8	(5, 3) 9	(5, 4) 10	(5, 5)	(5, 6)
6	(6, 1) 11	(6, 2) 12	(6, 3) 13	(6, 4) 14	(6, 5) 15	(6, 6)

7) *An even sum*

Shade every box whose sum is even: 2, 4, 6, 8, 10 or 12.

$$n(E) = 18$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

	1	2	3	4	5	6
1	(1, 1) 1	(1, 2)	(1, 3) 2	(1, 4)	(1, 5) 3	(1, 6)
2	(2, 1)	(2, 2) 4	(2, 3)	(2, 4) 5	(2, 5)	(2, 6) 6
3	(3, 1) 7	(3, 2)	(3, 3) 8	(3, 4)	(3, 5) 9	(3, 6)
4	(4, 1)	(4, 2) 10	(4, 3)	(4, 4) 11	(4, 5)	(4, 6) 12
5	(5, 1) 13	(5, 2)	(5, 3) 14	(5, 4)	(5, 5) 15	(5, 6)
6	(6, 1)	(6, 2) 16	(6, 3)	(6, 4) 17	(6, 5)	(6, 6) 18

8) *Both dice odd*

Shade every box where both numbers are odd: 1, 3 or 5.

$$n(E) = 3 \times 3 = 9$$

$$P(E) = \frac{n(E)}{n(S)} = \frac{9}{36} = \frac{1}{4}$$

	1	2	3	4	5	6
1	(1, 1) 1	(1, 2)	(1, 3) 2	(1, 4)	(1, 5) 3	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1) 4	(3, 2)	(3, 3) 5	(3, 4)	(3, 5) 6	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1) 7	(5, 2)	(5, 3) 8	(5, 4)	(5, 5) 9	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Q8

A couple has three children. Find the probability of:

1. All boys
2. All girls or all boys
3. Exactly two boys or exactly two girls
4. At least one child of each gender

Solution

Given: $S = \{BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG\}, \quad n(S) = 8$

1) *All boys*

$E = \{BBB\}, \quad n(E) = 1$

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{8}$$

2) *All girls or all boys*

$E = \{BBB, GGG\}, \quad n(E) = 2$

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{8} = \frac{1}{4}$$

3) *Exactly two boys or exactly two girls*

Two boys: BBG, BGB, GBB; two girls: BGG, GBG, GGB

$n(E) = 3 + 3 = 6$

$$P(E) = \frac{n(E)}{n(S)} = \frac{6}{8} = \frac{3}{4}$$

4) *At least one child of each gender*

Opposite: all the same gender = $\{BBB, GGG\}$

$P(E) = 1 - P(\text{all same})$

$$= 1 - \frac{2}{8} = \frac{3}{4}$$

Q9

A survey of 34 students at the Wall College of Business showed the following majors: Accounting 10, Finance 5, Economics 3, Management 6, Marketing 10. A student is selected at random.

1. What is the probability that he or she is a Management major?
2. Which approach to probability did you use?

Solution

Given: $n = 34$ students, Management = 6

1) *Probability of Management*

$$P(\text{Management}) = \frac{\text{Management}}{\text{total}} = \frac{6}{34} = \frac{3}{17} \approx 0.18$$

2) *Which approach?*

Empirical — *the probability comes from observed data (relative frequency)*

Q10

If $P(\overline{A}) = 3/8$, $P(B) = 3/5$ and $P(A \cap B) = 2/5$, find:

1. The occurrence of A
2. The non-occurrence of B
3. The occurrence of A or B
4. The non-occurrence of A or B $(\overline{A} \cup \overline{B})$
5. The occurrence of A only
6. The non-occurrence of A and B together $(\overline{A \cap B})$

Solution

Given: $P(\overline{A}) = \frac{3}{8}$, $P(B) = \frac{3}{5}$, $P(A \cap B) = \frac{2}{5}$

1) *The occurrence of A*

$$P(A) = 1 - P(\overline{A})$$

$$= 1 - \frac{3}{8} = \boxed{\frac{5}{8}}$$

2) *The non-occurrence of B*

$$P(\overline{B}) = 1 - P(B)$$

$$= 1 - \frac{3}{5} = \boxed{\frac{2}{5}}$$

3) *The occurrence of A or B*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{8} + \frac{3}{5} - \frac{2}{5} \quad P(A) = \tfrac{5}{8} \text{ from part 1}$$

$$= \frac{25}{40} + \frac{24}{40} - \frac{16}{40} = \boxed{\frac{33}{40}}$$

4) *The non-occurrence of A or B* $(\overline{A} \cup \overline{B})$

$$P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B)$$

$$= 1 - \frac{2}{5} = \boxed{\frac{3}{5}}$$

5) *The occurrence of A only*

$$P(A - B) = P(A) - P(A \cap B)$$

$$= \frac{5}{8} - \frac{2}{5} \quad P(A) = \tfrac{5}{8} \text{ from part 1}$$

$$= \frac{25}{40} - \frac{16}{40} = \boxed{\frac{9}{40}}$$

6) *The non-occurrence of A and B together* $(\overline{A \cap B})$

$$P(\overline{A \cap B}) = 1 - P(A \cap B)$$

$$= 1 - \frac{2}{5} = \boxed{\frac{3}{5}}$$

Q11

If $P(A \cap \overline{B}) = 1/12$, $P(B - A) = 5/12$ and $P(\overline{A} \cap \overline{B}) = 1/4$, find:

1. $P(A)$
2. $P(B)$
3. $P(A \cap B)$

Solution

Given: $P(A - B) = \frac{1}{12}$ (A only), $P(B - A) = \frac{5}{12}$ (B only), $P(\overline{A} \cap \overline{B}) = \frac{1}{4}$ (neither)

Start with part 3: we need $P(A \cap B)$ for parts 1 and 2.

3) $P(A \cap B)$

$$P(A \cup B) = 1 - P(\overline{A} \cap \overline{B})$$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(A \cap B) = P(A \cup B) - P(A - B) - P(B - A)$$

$$= \frac{3}{4} - \frac{1}{12} - \frac{5}{12} \quad P(A \cup B) = \tfrac{3}{4} \text{ from the line above}$$

$$= \frac{9}{12} - \frac{1}{12} - \frac{5}{12} = \frac{3}{12} = \frac{1}{4}$$

1) $P(A)$

$$P(A) = P(A - B) + P(A \cap B)$$

$$= \frac{1}{12} + \frac{3}{12} \quad \text{from part 3}$$

$$= \frac{4}{12} = \frac{1}{3}$$

2) $P(B)$

$$P(B) = P(B - A) + P(A \cap B)$$

$$= \frac{5}{12} + \frac{3}{12} \quad P(A \cap B) = \tfrac{1}{4} \text{ from part 3}$$

$$= \frac{8}{12} = \frac{2}{3}$$

Q12

If $P(A) = 0.5$, $P(B) = 0.6$, $P(C) = 0.4$, $P(A \cap B) = 0.3$, $P(A \cap C) = P(B \cap C) = 0.2$ and $P(A \cap B \cap C) = 0.1$, calculate:

1. $P(A \cup B)$
2. $P(A \cup (B \cap C))$
3. $P(A \cap (B \cup C))$
4. $P(A \cup B \cup C)$

Solution

Given: $P(A) = 0.5$, $P(B) = 0.6$, $P(C) = 0.4$, $P(A \cap B) = 0.3$, $P(A \cap C) = P(B \cap C) = 0.2$, $P(A \cap B \cap C) = 0.1$

1) $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.5 + 0.6 - 0.3 = 0.8$$

2) $P(A \cup (B \cap C))$

$$P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap B \cap C)$$

$$= 0.5 + 0.2 - 0.1 = 0.6$$

3) $P(A \cap (B \cup C))$

$$P(A \cap (B \cup C)) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$

$$= 0.3 + 0.2 - 0.1 = 0.4$$

4) $P(A \cup B \cup C)$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= 0.5 + 0.6 + 0.4 - 0.3 - 0.2 - 0.2 + 0.1$$

$$= 1.5 - 0.7 + 0.1 = 0.9$$

Q13

If $P(A) = 0.6$, $P(B) = 0.1$ and $P(A \cap B) = 0.02$, find the probability of:

1. A or B
2. B but not A

Solution

Given: $P(A) = 0.6$, $P(B) = 0.1$, $P(A \cap B) = 0.02$

1) *A or B*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.6 + 0.1 - 0.02 = 0.68$$

2) *B but not A*

$$P(B - A) = P(B) - P(A \cap B)$$

$$= 0.1 - 0.02 = 0.08$$